Capítulo 2. Teoría Cuántica de Campo

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Ejercicio Propuesto: Dado un campo escalar ϕ definido en 3 puntos $(\phi_1 \, , \phi_2 \, , \phi_3)$ y una magnitud C definida como: $C = -6\phi_1^2 - 6\phi_2^2 - 6\phi_3^2 - \sqrt{2}\phi_1\phi_2 - \sqrt{2}\phi_2\phi_3$.

Se pide:

1. Hallar la matriz A tal que:
   $$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix} (A) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = C$$
2. Diagonalizar A.
3. Demuestre que:
   $$C = -5\psi_1^2 - 6\psi_2^2 - 7\psi_3^2$$
   en donde hemos definido
   $$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

Solución.

1. Sea $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}=C$$

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \end{pmatrix}\begin{pmatrix} a_{11}\phi_1 & a_{12}\phi_2 & a_{13}\phi_3 \\ a_{21}\phi_1 & a_{22}\phi_2 & a_{23}\phi_3 \\ a_{31}\phi_1 & a_{32}\phi_2 & a_{33}\phi_3 \end{pmatrix}=C$$

$$a_{11}\phi_1^2+a_{12}\phi_1\phi_2+a_{13}\phi_1\phi_3+a_{21}\phi_1\phi_2+a_{22}\phi_2^2+a_{23}\phi_2\phi_3+a_{31}\phi_1\phi_3+a_{32}\phi_2\phi_3+a_{33}\phi_3^2=C$$

$$a_{11}\phi_1^2+a_{22}\phi_2^2+a_{33}\phi_3^2+\left(a_{12}+a_{21}\right)\phi_1\phi_2+\left(a_{13}+a_{31}\right)\phi_1\phi_3+\left(a_{23}+a_{32}\right)\phi_2\phi_3=$$
$$-6\phi_1^2-6\phi_2^2-6\phi_3^2-\sqrt{2}\phi_1\phi_2-\sqrt{2}\phi_2\phi_3$$

Igualamos los coeficientes.

$$a_{11}=-6$$
$$a_{22}=-6$$
$$a_{33}=-6$$
$$a_{12}+a_{21}=-\sqrt{2}\rightarrow-\frac{1}{\sqrt{2}}$$
$$a_{13}+a_{31}=0$$
$$a_{23}+a_{32}=-\sqrt{2}\rightarrow-\frac{1}{\sqrt{2}}$$

$$A=\begin{pmatrix} -6 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6 \end{pmatrix}$$

2. Diagonalizar A:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$A\vec{v}_2 = \lambda \vec{v}_2$$
$$A\vec{v}_3 = \lambda \vec{v}_3$$

$$\vec{v} = \begin{pmatrix} x_1 \\ y_2 \\ z_3 \end{pmatrix} \qquad A\begin{pmatrix} x_1 \\ y_2 \\ z_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ y_2 \\ z_3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ y_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda y_2 \\ \lambda z_3 \end{pmatrix}$$

$$\begin{pmatrix} -6x_1 & -\frac{1}{\sqrt{2}} y_2 & 0 \\ -\frac{1}{\sqrt{2}} x_1 & -6y_2 & -\frac{1}{\sqrt{2}} z_3 \\ 0 & -\frac{1}{\sqrt{2}} y_2 & -6z_3 \end{pmatrix} - \begin{pmatrix} \lambda x_1 \\ \lambda y_2 \\ \lambda z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -(6+\lambda)x_1 & -\frac{1}{\sqrt{2}} y_2 & 0 \\ -\frac{1}{\sqrt{2}} x_1 & -(6+\lambda)y_2 & -\frac{1}{\sqrt{2}} z_3 \\ 0 & -\frac{1}{\sqrt{2}} y_2 & -(6+\lambda)z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

De una ecuación matricial pasamos a un sistema de ecuaciones lineales

$$\begin{cases} -\left(6+\lambda\right)x_1 - \frac{1}{\sqrt{2}} y_2 = 0 \\ -\frac{1}{\sqrt{2}} x_1 - \left(6+\lambda\right) y_2 - \frac{1}{\sqrt{2}} z_3 = 0 \\ -\frac{1}{\sqrt{2}} y_2 - \left(6+\lambda\right) z_3 = 0 \end{cases}$$

Calcular el determinante:

$$\begin{vmatrix} -(6+\lambda) & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -(6+\lambda) & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -(6+\lambda) \end{vmatrix} = 0$$

$$-(6+\lambda)^3 + (6+\lambda) = 0$$

$$-(\lambda^3 + 18\lambda^2 + 108\lambda + 216) + (6+\lambda) = 0$$

$$-\lambda^3 - 18\lambda^2 - 107\lambda - 210 = 0$$

$$\lambda_1 = -5$$
$$\lambda_2 = -6$$
$$\lambda_3 = -7$$

Para $\lambda_1 = -5$

$$-(6-5)x_1 - \tfrac{1}{\sqrt{2}} y_2 = 0 \rightarrow -x_1 - \tfrac{1}{\sqrt{2}} y_2 = 0$$
$$-\tfrac{1}{\sqrt{2}} x_1 - (6-5) y_2 - \tfrac{1}{\sqrt{2}} z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} x_1 - y_2 - \tfrac{1}{\sqrt{2}} z_3 = 0$$
$$-\tfrac{1}{\sqrt{2}} y_2 - (6-5) z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} y_2 - z_3 = 0$$

$$x_1 = -\tfrac{1}{\sqrt{2}} y_2 \rightarrow x_1 = -\tfrac{1}{\sqrt{2}}\left(-\sqrt{2} z_3\right) = z_3$$
$$y_2 = -\sqrt{2} z_3$$

$x_1 = 1$, $z_3 = 1$ y $y_2 = -\sqrt{2}$

$$|\vec{v}| = \sqrt{(1)^2 + \left(-\sqrt{2}\right)^2 + (1)^2} = 2$$

$$\lambda_1 = -5 \rightarrow \vec{v}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Para $\lambda_2 = -6$

$$-(6-6)x_1 - \tfrac{1}{\sqrt{2}} y_2 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} y_2 = 0$$
$$-\tfrac{1}{\sqrt{2}} x_1 - (6-6) y_2 - \tfrac{1}{\sqrt{2}} z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} x_1 + -\tfrac{1}{\sqrt{2}} z_3 = 0$$
$$-\tfrac{1}{\sqrt{2}} y_2 - (6-6) z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} y_2 = 0$$

$$x_1 = -z_3$$
$$-\tfrac{1}{\sqrt{2}} y_2 = 0$$

$x_1 = 1$, $z_3 = -1$ y $y_2 = 0$

$$|\vec{v}| = \sqrt{(1)^2 + (0)^2 + (-1)^2} = \sqrt{2}$$

$$\lambda_2 = -6 \quad , \quad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Para $\lambda_3 = -7$

$$-(6-7)x_1 - \tfrac{1}{\sqrt{2}} y_2 = 0 \rightarrow x_1 - \tfrac{1}{\sqrt{2}} y_2 = 0$$
$$-\tfrac{1}{\sqrt{2}} x_1 - (6-7) y_2 - \tfrac{1}{\sqrt{2}} z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} x_1 + y_2 - \tfrac{1}{\sqrt{2}} z_3 = 0$$
$$-\tfrac{1}{\sqrt{2}} y_2 - (6-7) z_3 = 0 \rightarrow -\tfrac{1}{\sqrt{2}} y_2 + z_3 = 0$$

$$x_1 = \tfrac{1}{\sqrt{2}} y_2 \rightarrow x_1 = \tfrac{1}{\sqrt{2}} \left(\sqrt{2} z_3\right) = z_3$$
$$y_2 = \sqrt{2} z_3$$

$x_1 = 1$, $z_3 = 1$ y $y_2 = \sqrt{2}$

$$|\vec{v}| = \sqrt{(1)^2 + \left(\sqrt{2}\right)^2 + (1)^2} = 2$$

$$\lambda_3 = -7 \rightarrow \vec{v}_3 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Llamamos M a la matriz que tiene los vectores propios como columna:

$$M=\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

Para calcular la matriz A diagonalizada, $D=M^{T}AM$, podemos concluir que:

$$\begin{pmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -7 \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -6 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

3. Demostrar que: $C=-5\psi_1^2-6\psi_2^2-7\psi_3^2$
en donde hemos definido:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}=M\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2 \end{pmatrix}$$

Solución.

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}=\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}=\begin{pmatrix} \frac{1}{2}\psi_1+ & \frac{1}{\sqrt{2}}\psi_2+ & \frac{1}{2}\psi_3 \\ -\frac{1}{\sqrt{2}}\psi_1+ & 0+ & \frac{1}{\sqrt{2}}\psi_3 \\ \frac{1}{2}\psi_1- & \frac{1}{\sqrt{2}}\psi_2+ & \frac{1}{2}\psi_3 \end{pmatrix}$$

$$\phi_1=\tfrac{1}{2}\psi_1+\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3$$
$$\phi_2=-\tfrac{1}{\sqrt{2}}\psi_1+\tfrac{1}{\sqrt{2}}\psi_3$$
$$\phi_3=\tfrac{1}{2}\psi_1-\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3$$

$$C=-6\left(\tfrac{1}{2}\psi_1+\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3\right)^2-6\left(-\tfrac{1}{\sqrt{2}}\psi_1+\tfrac{1}{\sqrt{2}}\psi_3\right)^2-6\left(\tfrac{1}{2}\psi_1-\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3\right)^2-\sqrt{2}\left(\tfrac{1}{2}\psi_1+\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3\right)$$
$$\left(-\tfrac{1}{\sqrt{2}}\psi_1+\tfrac{1}{\sqrt{2}}\psi_3\right)-\sqrt{2}\left(-\tfrac{1}{\sqrt{2}}\psi_1+\tfrac{1}{\sqrt{2}}\psi_3\right)\left(\tfrac{1}{2}\psi_1+\tfrac{1}{\sqrt{2}}\psi_2+\tfrac{1}{2}\psi_3\right)$$

$$C=-6\left(\tfrac{1}{4}\psi_1^2+\tfrac{1}{2}\psi_2^2+\tfrac{1}{4}\psi_3^2+\tfrac{1}{\sqrt{2}}\psi_1\psi_2+\tfrac{1}{\sqrt{2}}\psi_2\psi_3+\tfrac{1}{2}\psi_1\psi_3\right)-6\left(\tfrac{1}{2}\psi_1^2-\psi_1\psi_3+\tfrac{1}{2}\psi_3^2\right)-$$
$$6\left(\tfrac{1}{4}\psi_1^2+\tfrac{1}{2}\psi_2^2+\tfrac{1}{4}\psi_3^2-\tfrac{1}{\sqrt{2}}\psi_1\psi_2-\tfrac{1}{\sqrt{2}}\psi_2\psi_3+\tfrac{1}{2}\psi_1\psi_3\right)-\sqrt{2}\left(-\tfrac{\sqrt{2}}{4}\psi_1^2-\tfrac{1}{2}\psi_1\psi_2+\tfrac{1}{2}\psi_2\psi_3+\tfrac{\sqrt{2}}{4}\psi_3^2\right)-$$
$$\sqrt{2}\left(-\tfrac{\sqrt{2}}{4}\psi_1^2+\tfrac{1}{2}\psi_1\psi_2-\tfrac{1}{2}\psi_2\psi_3+\tfrac{\sqrt{2}}{4}\psi_3^2\right)$$

$$C=-\tfrac{3}{2}\psi_1^2-3\psi_2^2-\tfrac{3}{2}\psi_3^2-3\sqrt{2}\psi_1\psi_2-3\sqrt{2}\psi_2\psi_3-3\psi_1\psi_3-3\psi_1^2+6\psi_1\psi_3-3\psi_3^2-\tfrac{3}{2}\psi_1^2-3\psi_2^2-\tfrac{3}{2}\psi_3^2+$$
$$3\sqrt{2}\psi_1\psi_2+3\sqrt{2}\psi_2\psi_3-3\psi_1\psi_3+\tfrac{1}{2}\psi_1^2+\tfrac{1}{\sqrt{2}}\psi_1\psi_2-\tfrac{1}{\sqrt{2}}\psi_2\psi_3-\tfrac{1}{2}\psi_3^2+\tfrac{1}{2}\psi_1^2-\tfrac{1}{\sqrt{2}}\psi_1\psi_2+\tfrac{1}{\sqrt{2}}\psi_2\psi_3-\tfrac{1}{2}\psi_3^2$$

Simplificando la expresión queda demostrado:

$$C=-5\psi_1^2-6\psi_2^2-7\psi_3^2$$